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TOPICAL ISSUE

Optimization of Interception Plan for Rectilinearly Moving Targets

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Abstract—The article considers the problem of combinatorial optimization of interception of rectilinearly moving targets as a modification of the traveling salesman problem. New macro characteristics and definitions for this problem are introduced and used to classify the obtained solutions. Vector criteria composed of several important for applications functionals are described. The principles of no-waiting and maximum velocity are proved for two types of criteria. An intelligent brute-force algorithm with dynamic programming elements for finding optimal plans according to the introduced intercept criteria is proposed and implemented. Statistics of solutions of the developed algorithm is collected for a set of different initial parameters and the proposed macro characteristics are investigated. The conclusions about their applicability as local rules for the greedy algorithm for finding a suboptimal intercept plan are drawn.

Keywords: moving targets traveling salesman problem, combinatorial optimization, simple motion model

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1. INTRODUCTION

A recent development of intelligent technologies in the field of unmanned autonomous vehicles makes it possible to use them cooperatively in a vast variety of applications and scenarios that were considered impossible before. One such application is the problem of preventing moving targets from infiltration of a given point in space by intercepting each one. The problem of optimal choice of the traverse order of targets is found to be crucial in the formalization and further solution of the problem of intercepting a set of targets. An optimal choice in terms of one criterion may turn out to be poor when another criterion is used. For example, if the optimal choice of the traverse order of moving targets is related to the requirement that targets should be intercepted in the shortest possible time at the greatest possible distance from the defended point, then an internal contradiction of such requirement can be shown. Let us consider a situation in which one defender faces two enemy targets, one of them is fast and the other is slow, and the defender's starting point is attacked from diametrically opposite directions. For the fastest execution time it is needed to intercept the slow target first, letting the fast target get closer, but if the fast target is intercepted first, then the shortest distance to the defended point will be greater than in the first case.

The Moving Targets Traveling Salesman Problem (MTTSP) [1–4] is the closest problem statement to the problem of constructing an optimal plan for intercepting moving targets studied in this paper. The MTTSP is a generalization of the traveling salesman problem (TSP). In 1972, the NP-completeness of the Hamiltonian cycle problem was shown to imply NP-completeness of the

MTTSP [5]. One of the first MTTSP statements for rectilinearly moving targets was in paper [1] where it was found that the dynamic programming apparatus could be used to construct an efficient algorithm for finding the optimal intercept plan.

In MTTSP, it is generally assumed that the control object movement is subject to simple motion model (the controlled input is a velocity vector). For some conditions [1, 4] such an assumption makes it possible to switch from a discrete-continuous optimization problem to a discrete optimization problem. The model where the controlled input is a velocity vector can be a sufficiently rough approximation for constructing reference trajectories of a real control object, but using more accurate models that take into account, for example, the maneuverability of the control object, do not allow to switch from a discrete-continuous problem to a discrete one to obtain an accurate solution even in the case of stationary targets [6, 7]. In this case, if the control object is sufficiently maneuverable and the distances between targets are large (in comparison to the minimum turning radius of the object), then taking into account maneuverability in the problem of target traverse planning does not affect the structure of the optimal plan.

Methods for solving MTTSP can be categorized as follows:

- with time sampling [3, 8, 9] or without [1, 4, 8, 10, 11];
- giving an optimal solution [3, 4, 8] or suboptimal [9–11];
- deterministic [1, 3, 4, 8, 9, 12] or random [10, 11, 13].

In [4] an algorithm for constructing a guaranteed intercept plan based on the notion of target danger is proposed for the problem of preventing intrusion of targets into a given point with the traveling salesman (TS) returning to it after each encounter with the target.

This paper considers the problem of optimizing an intercept plan for a vector criteria. The concepts of danger, convenience and complexity of intercept are formalized. The preference is given to deterministic methods without time discretization that give an optimal solution (without guarantees of fast completion).

The paper consists of an introduction, four sections and a conclusion and has the following structure. In Section 2, a new formulation of the problem of finding an optimal plan for intercepting targets moving rectilinearly to one protected point is formalized, the set of acceptable plans, vector criteria of the problem are introduced, and the definition of a guaranteed intercept plan is given. In Section 3, the theorems of guaranteed intercept and the principle of non-optimality of no-waiting are proved, and new notions of danger, convenience and complexity of intercept plan are introduced. Section 4 is devoted to an intelligent algorithm for brute-force search that significantly reduces the number of computations. Later in 5 the results of simulation based on the proposed algorithm are presented and the properties of the optimal plans are statistically investigated. In the 6 plans for further work are presented.

2. PROBLEM STATEMENT

2.1. Mathematical Model

Let us assume that the protected point is located on the plane at the coordinate origin. The targets appear on the outer boundary of a circle of radius R in a layer of width $2\Delta R$ in a sector with a central angle α. Targets move rectilinearly with known velocities of given range $[v_{\min}, v_{\max}]$. At the initial moment TS is situated at the origin and it controlled with velocity $v(t) \in [0, V]$, $V > v_{\max}$. It is assumed that moving targets must be serviced by the salesman as far from the protected object as possible, minimizing the danger as much as possible.

Definition 1. The set of initial conditions for targets and TS with all parameters of the problem being fixed is called the initial state.

The state may change as the current data about the objects is changed and refined. Since any moment when all parameters of the problem are known can be chosen as the initial moment, the situation at this moment will be called the current state.

Let us assume that there are only m targets and each of them is located at the initial moment at the point $\boldsymbol{r}_j^0 = (x_j^0, y_j^0)$, where $j = 1, \ldots, m$. Each target moves with constant velocity $\boldsymbol{v}_j = (v_{x,j}, v_{y,j})$. Thus, the trajectory of each target is a straight line

$$\mathbf{r}_j(t) = \boldsymbol{r}_j^0 + \boldsymbol{v}_j t, \quad j = 1, \ldots, m \tag{1}$$

with constrained parameters

$$\begin{gathered} ||\boldsymbol{v}_j|| \in [v_{\min}, v_{\max}], \quad v_{\max} < V, \\ ||\boldsymbol{r}_j^0|| \in [R - \Delta R, R + \Delta R], \\ \arctan \frac{y_j^0}{x_j^0} \in \left[\frac{\pi}{2} - \frac{\alpha}{2}, \frac{\pi}{2} + \frac{\alpha}{2}\right]. \end{gathered} \tag{2}$$

When the target reaches the origin, it is meaningless to service it. This moment in time for target number j can be calculated as follows:

$$t_j^0 = \frac{\|\boldsymbol{r}_j^0\|}{\|\boldsymbol{v}_j\|}. \tag{3}$$

The dynamics of TS is described with a system of differential equations of the following form:

$$\begin{aligned} \dot{\mathrm{x}}^I(t) &= v(t) \cos \psi(t), \quad v(t) \in [0, V]; \\ \dot{\mathrm{y}}^I(t) &= v(t) \sin \psi(t), \quad \psi(t) \in [0, 2\pi); \end{aligned} \tag{4}$$

where $\mathbf{r}^I(t) = (\mathrm{x}^I(t), \mathrm{y}^I(t))$ is the position of TS at the moment t; $\psi(t)$ is the velocity direction control on the plane. The salesman is at the origin at the initial moment $\mathbf{r}^I(0) = (\mathrm{x}^I(0), \mathrm{y}^I(0)) = (0, 0)$.

In order to formulate the plan construction problem as an optimization problem, it is necessary to give a formal description of the problem model, which include definitions of problem solution, its acceptability, and a quality criterion of the problem.

The principles of non-optimality of no-waiting and maximum velocity motion is proved later in the paper. The intercept function is constructed for the problem of the fastest intercept of a target moving uniformly along a straight line by TS whose movement is subject to simple motion model, and is reduced to finding the smallest positive root of the following quadratic equation with respect to the intercept time τ:

$$(\boldsymbol{r}_j + \boldsymbol{v}_j \tau)^2 = V^2 \tau^2.$$

Here $\boldsymbol{r}_j = \mathbf{r}_j(t) - \mathbf{r}^I(t)$ is a vector of relative positions of TS and the target with number j, $\mathbf{r}^I(t)$ is a current position of TS, V is a maximum velocity of TS, $\boldsymbol{v}_j$ is a velocity vector of the target. Let us denote the smallest non-negative root of this equation by $\tau(\boldsymbol{r}_j, \boldsymbol{v}_j)$. It can be shown that when $\boldsymbol{v}_j^2 < V^2$ the following expression is valid

$$\tau(\boldsymbol{r}_j, \boldsymbol{v}_j) = \frac{(\boldsymbol{v}_j, \boldsymbol{r}_j) + \sqrt{(\boldsymbol{v}_j, \boldsymbol{r}_j)^2 + \boldsymbol{r}_j^2 (V^2 - \boldsymbol{v}_j^2)}}{V^2 - \boldsymbol{v}_j^2}. \tag{5}$$

2.2. Individual Plan

An individual plan π for TS intercepting $k \in \{0, \ldots, m\}$ targets is a tuple of k different numbers from $\mathcal{M} = \{1, \ldots, m\}$. The order of the elements in the tuple determines the order according to which the targets are intercepted. The space of all individual plans intercepting $k \in \{0, \ldots, m\}$ targets can be described as follows:

$$\Pi_k = \left\{ (\pi_1, \ldots, \pi_k) \in \mathcal{M}^k : \ \forall p, q \in \{1, \ldots, k\} \quad p \neq q \to \pi_p \neq \pi_q \right\}.$$

Let us consider for example $m = 2$:

$$\begin{aligned} &\Pi_0 = \{()\}, \\ &\Pi_1 = \{(1,), (2,)\}, \\ &\Pi_2 = \{(1,2), (2,1)\}. \end{aligned}$$

$()$ denotes an empty tuple (an individual plan prescribing inaction). Thus, if an individual plan $\pi = (\pi_1, \ldots, \pi_k) \in \Pi_k$ is prescribed for TS, then TS should firstly intercept the target with number π_1, then – the target with number π_2, and so on.

The space of all plans for a given number of targets m is the following set:

$$\Pi = \bigcup_{k=0}^{m} \Pi_k.$$

First, let us compute the minimum time $T(\pi)$ that it takes a salesman to execute an individual plan $\pi = (\pi_1, \ldots, \pi_k)$. Using the definition of $\tau(\boldsymbol{r}_j, \boldsymbol{v}_j)$ from (5), the following recursive expression can be obtained:

$$T(\pi) = \begin{cases} 0, & k = 0; \\ \tau(\boldsymbol{r}^0_{\pi_1}, \boldsymbol{v}_{\pi_1}), & k = 1; \\ t + \tau(\mathbf{r}_{\pi_k}(t) - \mathbf{r}^I(t), \boldsymbol{v}_{\pi_k}), & k > 1, \ \text{here } t = T((\pi_1, \ldots, \pi_{k-1})). \end{cases} \tag{6}$$

Let us also write out the constraint that every target entering the plan must be intercepted on time, i.e. before reaching the coordinate origin:

$$\mathrm{OnTime}(\pi) = \left(\forall j \in \{1, \ldots, k\} : \ T((\pi_1, \ldots, \pi_j)) \leqslant t_{\pi_j} \right).$$

It should be noted that last restriction can be checked recurrently. If for an individual plan $\pi = (\pi_1, \ldots, \pi_k)$ the corresponding constraint is satisfied and the target j is not considered in the individual plan π, then the following expression can be used to check the constraint for the individual plan $\pi + j = (\pi_1, \ldots, \pi_k, j)$:

$$\mathrm{OnTime}(\pi + j) = \mathrm{OnTime}(\pi) \ \& \ T(\pi + j) \leqslant t_j.$$

Definition 2. The interception plan is acceptable if it allows to intercept moving targets on time. The set of acceptable plans is the following

$$\Pi_A = \{\pi \in \Pi : \ \mathrm{OnTime}(\pi)\}.$$

2.3. Criteria and Definitions

The criterion of the problem is associated with the loss functional J. Obviously, a lesser loss function corresponds to a better solution. The loss functional must be defined on the set of acceptable plans Π_A, i.e. for each plan an estimate of the losses can be made. The optimal solution to the traverse plan construction problem is the plan $\pi^* \in \Pi_A$ that minimizes the value of the loss functional:

$$\pi^* \in \arg \min_{\pi \in \Pi_A} J[\pi]. \tag{7}$$

The set of acceptable plans Π_A is finite and contains at least one element (which is the empty plan), so the minimization problem always has a solution, maybe not the only one. The equal sign in the expression $\pi^* = \arg \min\limits_{\pi \in \Pi_A} J[\pi]$ means that solution is unique.

Basic functionals that can be used to build the criterion of the problem are

- **Missed targets** (the number of missed targets that reached the origin). All targets that were not included in the individual plan $\pi \in \Pi_A$ will reach the origin, i.e. the number of missed targets is calculated as follows

$$n_0[\pi] = m - \mathrm{card}(\pi),$$

  where $\mathrm{card}(\pi)$ is the length of the plan π.
- **Execution time.** It is calculated as the execution time of the individual plan

$$T_{\mathrm{sum}}[\pi] = T(\pi).$$

- **Minimum interception distance.** The minimum distance from the origin to target interception point for the plan $\pi \in \Pi_A$ is calculated as follows

$$D_{\min}[\pi] = \min_{j \in \{1,\dots,m\}} \|\mathbf{r}_{\pi_j}(T((\pi_1, \dots, \pi_j)))\|.$$

  If the plan π is empty, it will be formally assumed that $D_{\min}[\pi] = 0$.

Not all of the mentioned functionals are suitable for the role of the problem criterion. Indeed, minimizing only the execution time of the plan leads to an empty plan consisting in inaction, and it will be optimal because zero time units are required for its execution. Only a loss functional, describing the number of targets that reached the origin, can be used as problem criterion.

Definition 3. The interception plan is guaranteed if $n_0[\pi] = 0$. The set of guaranteed plans is denoted by Π_G.

Some generalization of the comparison method is needed to adequately compare the listed functionals in the final problem criterion. Most of the mentioned functionals make sense in the minimization problem if they are combined together to form a criterion. For example, if two plans are compared primarily on the number of missed targets that reached the origin, and secondarily, on the total interception time, then such a combined loss functional adequately capture the essence of the point defense problem. In other words, if some plan π_1 admits skipping one target to the origin and the execution time of the plan is 7, i.e. $n_0[\pi_1] = 1$ and $T_{\mathrm{sum}}[\pi_1] = 7$, and a plan π_2 admits skipping one target to the origin and the execution time of the plan is 8, i.e. $n_0[\pi_2] = 1$ and $T_{\mathrm{sum}}[\pi_2] = 8$, then plan π_1 is better than plan π_2, i.e. the tuples $(1, 7) < (1, 8)$ can be formally compared. This comparison is similar to the positional comparison of real numbers, where each digit at the corresponding position of a real number is compared to the corresponding digit of another number until no differences in values are found from left to right. Let us formalize the above on the concept of lexicographic order.

Definition 4. A tuple of numbers $\boldsymbol{a} = (a_1, a_2, \ldots, a_p)$ is less than a tuple of numbers $\boldsymbol{b} = (b_1, b_2, \ldots, b_q)$ if there exists a number $k \in \{1, \ldots, \min(p, q)\}$ such that $a_i = b_i$ for $i < k$ and $a_k < b_k$. If for all $k \in \{1, \ldots, \min(p, q)\}$ $a_k = b_k$, then for $p < q$ it is assumed that $\boldsymbol{a} < \boldsymbol{b}$. In other cases, it is assumed that $\boldsymbol{a} \geqslant \boldsymbol{b}$.

Examples:

$$(1,2) < (1,3), \quad (0,1) < (1,2), \quad (1,2) < (1,2,1), \quad () < (1,2), \quad (1,2) < (2,).$$

The main criteria of the target interception problem for an arbitrary acceptable plan $\pi \in \Pi_A$ are formulated using the definition of tuple comparison.

- **Missed targets + Execution Time.** The quality criterion for the obtained plans is the following

$$J_T[\pi] = (n_0[\pi], T_{\text{sum}}[\pi]). \tag{8}$$

  Minimizing of this loss functional is primarily aimed at minimizing the number of missed targets and time execution of the plan secondarily.
- **Missed targets + Minimum interception distance.** The quality criterion is the following

$$J_D[\pi] = (n_0[\pi], -D_{\min}[\pi]). \tag{9}$$

  Minimization of this loss functional is primarily aimed at minimizing the number of missed targets and secondarily at maximizing the distance of the closest target approaching to the origin.
- **Missed Targets + Minimum interception distance + Execution Time.** The quality criterion is the following

$$J_{DT}[\pi] = (n_0[\pi], -D_{\min}[\pi], T_{\text{sum}}[\pi]). \tag{10}$$

  Minimization of this loss functional is primarily aimed at minimizing the number of missed targets, secondarily at maximizing the distance of the closest target approaching to the origin and thirdly at minimizing of total execution time.

Let us formulate the optimization problem.

Problem 1. For m targets moving along trajectories (1) with constraints on the motion parameters (2), it is required to find an optimal according to criterion (8) or (10) intercept plan $\pi \in \Pi_A$ for TS with dynamics (4).

3. PROPERTIES OF THE OPTIMAL INTERCEPT PLAN SEARCH PROBLEM

The following definitions and terms are needed to describe the problem properties.

Using the formula (5), the time $\tau_j(t) = \tau(\boldsymbol{r}_j, \boldsymbol{v}_j)$ of intercepting the jth target from the current state and the time $t_j(t)$ of movement of the jth target to the coordinate origin are introduced.

Definition 5. A danger K_j of the jth target is defined as the inverse value of the movement time needed to reach the coordinate origin, namely

$$K_j(t) = \frac{1}{t_j(t)}.$$

The danger is a property of the target. The less time before the target enters the protected region, the more dangerous it is considered.

Definition 6. The convenience U_j of intercepting the jth target is the inverse of the time which TS needs to intercept this target from the current state, namely

$$U_j(t) = \frac{1}{\tau_j(t)}.$$

Convenience is a property of TS's action with respect to the target. The less time it takes, the more convenient it is to intercept the target.

Definition 7. The intercept complexity $C[\pi]$ of the plan π is the maximum time between two consecutive intercepts in the plan, namely

$$C[\pi] = \max_{\substack{\{\pi_j\} \in \pi, \\ 1 < j \leqslant m.}} \tau_{\pi_j}(T((\pi_1, \dots, \pi_{j-1}))). \tag{11}$$

Complexity is a property of TS's plan. The less time there is between two consecutive target intercepts during plan execution, the less complex it is.

Definition 8. The average complexity $\widehat{C}[\pi]$ of an intercept plan π is the average time between two consecutive intercepts in the plan, namely

$$\widehat{C}[\pi] = \frac{1}{m-1} \sum_{\substack{\{\pi_j\} \in \pi, \\ 1 < j \leqslant m.}} \tau_{\pi_j}(T((\pi_1, \dots, \pi_{j-1}))). \tag{12}$$

Average complexity characterises the durations between consecutive interceptions in a plan. If all the targets are intercepted consecutively without long interceptions then this plan is less complex in average compared to the plan containing several long interceptions.

The danger is directly related to the criteria for execution of the intercept plan, while convenience and complexity relate to the sequential selection of the next target and the quality of plan execution according to the time-optimal criterion. If there is a target traverse plan where the consecutive intercepts occur as conveniently as possible and there is no target miss, then the execution time of the plan is often close to optimal.

The notions of danger and convenience can be generalised for the current state.

Definition 9. The danger of the current state is a decreasingly ordered tuple of m target danger values

$$(K_{j_1}(t), \dots, K_{j_m}(t)). \tag{13}$$

The order of targets in the danger tuple does not change during the execution of the plan.

Definition 10. The convenience of the current state is a decreasingly ordered tuple of m target convenience values

$$(U_{j_1}(t), \dots, U_{j_m}(t)).$$

The convenience of the current state depends on the position of TS and changes over time.

When targets are intercepted, tuple lengths are reduced. The complexity of the plan is directly related to the convenience of the traverse. The optimal plan combines all of the above state characteristics.

Theorem 1. *For any initial state and any number of targets in the problem 1, there is a guaranteed intercept plan* $\pi \in \Pi_G$.

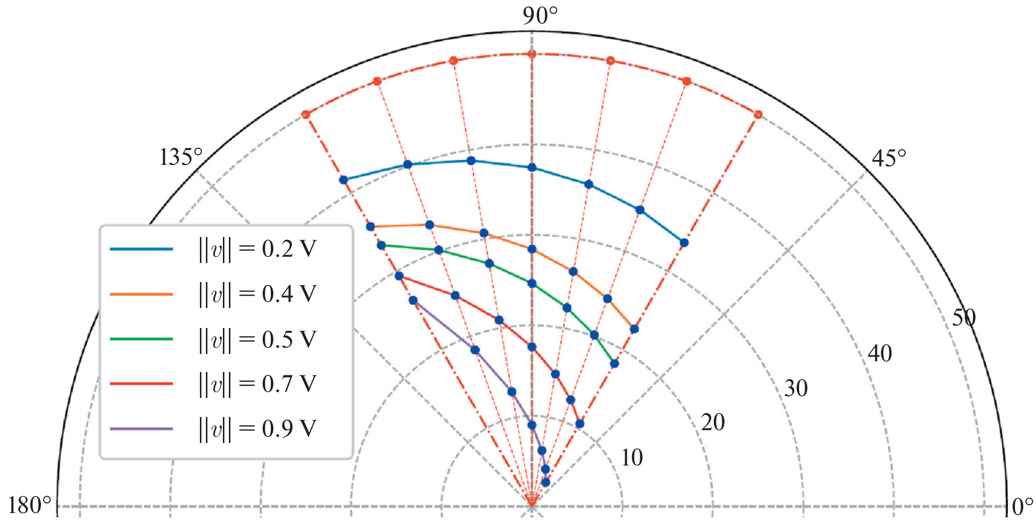

Fig. 1. Intercept targets with velocities $||\boldsymbol{v}|| = \{0.2V, 0.4V, 0.5V, 0.7V, 0.9V\}$, located on the boundary of a circular sector with a centre angle $\alpha = 60°$.

Proof. It is possible to carry out the proof using Theorem 10 of [4], but then the features of the problem will be left out.

The intercept plan is created according to the initial state danger (13) calculated similarly to [4]. Let us prove that such a plan is guaranteed.

If in the initial state the distance $||\boldsymbol{r}_j^0||$, where the index j corresponds to the most dangerous target, is not the minimum among all $||\boldsymbol{r}_k^0||$, $k = 1, \dots, j-1, j+1, \dots, m$, then the moment of the start of TS's movement is postponed. Then a radius R_0 is found such that the targets cross it in decreasing order of danger $(K_{j_1}, \dots, K_{j_m})$ by permutation of the targets $(j_1, \dots, j_m)$ with respect to the initial numbering $(1, \dots, m)$. When the most dangerous target is intercepted, all others are outside the radius of the current interception. Due to the superiority of the velocity of TS, no target will reach the origin, which is shown in the example of a situation where the next most dangerous target j_{d+1} is located diametrically opposite to the current one $||\mathbf{r}_{j_d}(t)|| < ||\mathbf{r}_{j_{d+1}}(t)||$, $t = T((j_1, \dots, j_d))$. In this case, the difference between the times it takes the target and TS to reach the origin is equal to

$$\frac{||\boldsymbol{r}_{j_{d+1}}(t)||}{||\boldsymbol{v}_{j_{d+1}}||} - \frac{||\boldsymbol{r}_{j_d}(t)||}{V} = \frac{V \cdot ||\boldsymbol{r}_{j_{d+1}}(t)|| - ||\boldsymbol{v}_{j_{d+1}}|| \cdot ||\boldsymbol{r}_{j_d}(t)||}{V \cdot ||\boldsymbol{v}_{j_{d+1}}||} > 0.$$

This means that in an extreme case, when the interception occurs along the beams of one straight line, TS will have time to get to the origin, after which TS will intercept the next target. In cases where the interception is carried out on the remaining beams, it is obvious that the targets also will not reach the origin. This proves that the danger interception plan is guaranteed. □

Example 1. Let the targets be uniformly distributed on the boundary of a circular sector with a radius R and a central angle α and move with equal velocities $||\boldsymbol{v}||$. Then the optimal intercept according to the criteria $J_T[\pi]$ and $J_{DT}[\pi]$ is carried out along a trajectory close to the logarithmic spiral [14] following the plan π on which $n_0[\pi] = 0$, as shown in Fig. 1. In this example, the danger and convenience of the initial setting are $(K_1, \dots, K_m) = (U_1, \dots, U_m) = (||\boldsymbol{v}||/R, \dots, ||\boldsymbol{v}||/R)$ and do not allow to make an initial choice of target. Once the rightmost or leftmost target in a sector has been intercepted, the remaining targets will also be equally distributed in danger. However, the intercept convenience tuple will not only have an order of the remaining targets, but also this order will not change after each intercept.

Theorem 2. *For criteria $J_T[\pi]$ and $J_{DT}[\pi]$ in problem 1 the following principles are valid:*

1) the no-waiting principle (on the optimal plan TS cannot be motionless),

2) the principle of maximum velocity (on the optimal plan TS moves at the maximum possible velocity).

Proof. The guaranteed intercept ($n_0[\pi] = 0$) minimising the number of missed targets for criteria $J_T[\pi] = (n_0[\pi], T_{\text{sum}}[\pi])$ and $J_{DT}[\pi] = (n_0[\pi], -D_{\min}[\pi], T_{\text{sum}}[\pi])$, can be obtained by the Theorem 1 by choosing the plan $\pi = (i_1, \ldots, i_m)$ according to the initial state danger $(K_{i_1}, \ldots, K_{i_m})$.

Further optimisation of vector criteria $J_T[\pi]$ and $J_{DT}[\pi]$ is performed according to guaranteed plans ($n_0[\pi] = 0$), consisting of at least one plan $\pi = (i_1, \ldots, i_m)$.

The validity of principles of no waiting and maximum velocity when minimising $T_{\text{sum}}[\pi]$ is shown in Lemma 1 of [4], which finishes the proof of the theorem for the functional $J_T[\pi]$.

Maximizing the functional $D_{\min}[\pi]$ in the criterion J_{DT} for a finite number of guaranteed plans leads to finding the plan π^*. Let us fix this plan and find the first target number j in the plan, on which the minimum distance to the origin is reached. The plan $\pi^* = (\pi_{1,j}, \pi_{j,m})$ is divided into two parts: $\pi_{1,j}$ before the goal j inclusive and $\pi_{j,m}$ after the goal j, then $D_{\min}[\pi^*] = D_{\min}[\pi_{1,j}]$. Increasing the execution time of part of the plan $\pi_{1,j}$ by waiting time or moving at less than maximum velocity leads to a decrease $D_{\min}[\pi_{1,j}]$ similar to Lemma 1 of [4]. Further, the part of the plan $\pi_{j,m}$ is optimal in execution time by Bellman's principle of optimality. The part $T_{\text{sum}}[\pi]$ of the criterion after reaching a minimum on the functional $D_{\min}[\pi]$ is responsible for this. Therefore, waiting and slowing down is impossible for TS on $\pi_{j,m}$ and hence on the whole π^*. $\square$

4. OPTIMAL INTERCEPT PLAN FINDING ALGORITHM

The algorithm for constructing a traverse plan for a single TS and many moving targets is based on brute-force sorting of plans with an initial sorting of targets by danger and an intelligent rule for discarding obviously non-optimal branches of the search in the process of its operation. It is guaranteed that the algorithm finds the optimal intercept plan.

To introduce some important notions that are necessary to understand the algorithm, let us first consider the simplest case of a brute-force search. The work of the algorithm in this case can be illustrated by the transition matrices in Table 1, which show the complete sequence of plans considered during the operation of the algorithm. Table 1 is called the plan search table. The criterion of the problem in the algorithm is from (10):

$$J_{DT}[\pi] = (n_0[\pi], -D_{\min}[\pi], T_{\text{sum}}[\pi]).$$

In Table 1 the transition matrices are numbered from 1 to $m! = 24$. For each matrix, a vector of indices is written in the column on the left that defines the intercept plan. The resulting plan is a sequence of marked circles in the matrix according to the index vector in ascending order of row number.

The uppermost index in the column is marked with an asterisk, from which a new calculation of the next plan begins. An intermediate state, characterised by a part of the already computed plan, is stored to save computational resources.

Statement 1. *The total number of calls of the single intercept function* (5) *in a brute force search of all variants in the algorithm with intermediate saving of calculations is described by the recurrence formula*

$$f(m) = m(f(m-1) + 1). \tag{14}$$

Table 1. Plan search table consisting of transition matrices for the case of brute-force search with $m = 4$

1

1	1	2	3	4
1	2	3	4	
1	3	4		
1	4			

2

1	1	2	3	4
1	2	3	4	
*2	3	4		
1	3			

3

1	1	2	3	4
*2	2	3	4	
1	2	4		
1	4			

4

1	1	2	3	4
2	2	3	4	
*2	2	4		
1	2			

5

1	1	2	3	4
*3	2	3	4	
1	2	3		
1	3			

6

1	1	2	3	4
3	2	3	4	
*2	2	3		
1	2			

7

*2	1	2	3	4
1	1	3	4	
1	3	4		
1	4			

8

2	1	2	3	4
1	1	3	4	
*2	3	4		
1	3			

9

2	1	2	3	4
*2	1	3	4	
1	1	4		
1	4			

10

2	1	2	3	4
2	1	3	4	
*2	1	4		
1	1			

11

2	1	2	3	4
*3	1	3	4	
1	1	3		
1	3			

12

2	1	2	3	4
3	1	3	4	
*2	1	3		
1	1			

13

*3	1	2	3	4
1	1	2	4	
1	2	4		
1	4			

14

3	1	2	3	4
1	1	2	4	
*2	2	4		
1	2			

15

3	1	2	3	4
*2	1	2	4	
1	1	4		
1	4			

16

3	1	2	3	4
2	1	2	4	
*2	1	4		
	1			

17

3	1	2	3	4
*3	1	2	4	
1	1	2		
1	2			

18

3	1	2	3	4
3	1	2	4	
*2	1	2		
1	1			

19

*4	1	2	3	4
1	1	2	3	
1	2	3		
1	3			

20

4	1	2	3	4
1	1	2	3	
*2	2	3		
1	2			

21

4	1	2	3	4
*2	1	2	3	
1	1	3		
1	2			

22

4	1	2	3	4
2	1	2	3	
*2	1	3		
1	1			

23

4	1	2	3	4
*3	1	2	3	
1	1	2		
1	2			

24

4	1	2	3	4
3	1	2	3	
*2	1	2		
1	1			

Thus, the number of calls of the single intercept function is significantly reduced. Example with $m = 4$:

$$f(m) = 64, \tag{15}$$

whereas for the case of brute force search the number of calls of this function is $F(m) = m! \cdot m = 96$. For a larger number of targets $m = 10$ respectively there are

$$\begin{aligned} f(m) &= 9\,864\,100, \\ F(m) &= 36\,288\,000. \end{aligned} \tag{16}$$

Saving of the current state of the plan already significantly reduces the number of computations. However, the main gain in the efficiency of the proposed algorithm is due to its problem orientation specifics and the possibility of discarding non-optimal chains of plans, whose criterion values is worse than ones of the currently saved plan. The algorithm consists of the following sequence of actions.

Algorithm 1. Finding a traverse plan.

(1) Targets are sorted by danger K_i.
(2) An auxiliary search matrix (matrix with index 1 in Table 1) is filled in and used to form a sequence of the plans.
(3) At each new step of the algorithm, the transition through the states of the full plan search table $1 \dots m!$ (Table 1) is performed according to the criterion (10).
(4) The first case: an intercept plan for all targets has not yet been found. In this case:
   (a) The transition in the plan search table is performed according to the parameter of the number of intercepted targets.
   (b) If the next plan is impossible to complete (the target reaches the origin) and the new considered plan has the same number of intercepted targets, the distance and time criteria are checked and the best plan is stored in memory.
   (c) The branch may be discarded if the number of targets missed at the origin has become worse with respect to the saved plan.
(5) The second case: if a plan that intercepts all targets is found. Then:
   (a) Any missed target in the new plan leads to the end of consideration of the current chain of plans.
   (b) If the new plan intercepts all targets, the distance and time criteria are checked and the better of the two plans is stored in memory.
(6) The last saved plan is the optimal plan.

The proposed initial sorting of targets by danger is used to discard non-optimal plan chains in the early stages of the Algorithm 1.

5. MODELLING AND RESULTS DISCUSSION

The interception Algorithm 1 was implemented in Matlab using the functions (5) and (6). Modelling has shown that the running time of the algorithm is acceptable for real-time applications and is strongly reduced relative to the brute-force algorithm. For 1000 experiments, the running time was 200 s, which means that the average running time for one initial state is 0.2 s.

1000 different initial states are considered, for which the following basic parameters are chosen:

- The number of targets is $m = 15$.
- The central angle of the sector where the targets are located is $\alpha = 60°$.
- The values $||\mathbf{r}_j||, j = 1, \dots, m$ are uniformly distributed in $[800,\ 1000]$.
- Target velocities are uniformly distributed in $[0.5V,\ 0.7V]$.

For each state, the danger, convenience, and optimal traverse plans are found according to the criteria $J_T[\pi]$ and $J_{DT}[\pi]$ using the Algorithm 1. Tables 2 and 3 give statistics on how often the first few objectives of the optimal plan turn out to be the most dangerous/convenient.

Tables 2 and 3 show that the statistics of selecting the first target in the plan differs from the statistics in the following steps, since the initial state is significantly different from the state that arise after each interception. In more than 70% of cases, according to the obtained statistics, the first target of the optimal plan matches with the most dangerous or the most convenient target, which can be used to construct greedy algorithms based on local rules according to danger or convenience instead of brute-force algorithms.

It was found for 1000 initial settings that in 24.6% of cases, the optimal plan π^* by criterion $J_{DT}[\pi^*]$ coincides with the optimal plan by criterion $J_T[\pi^*]$.

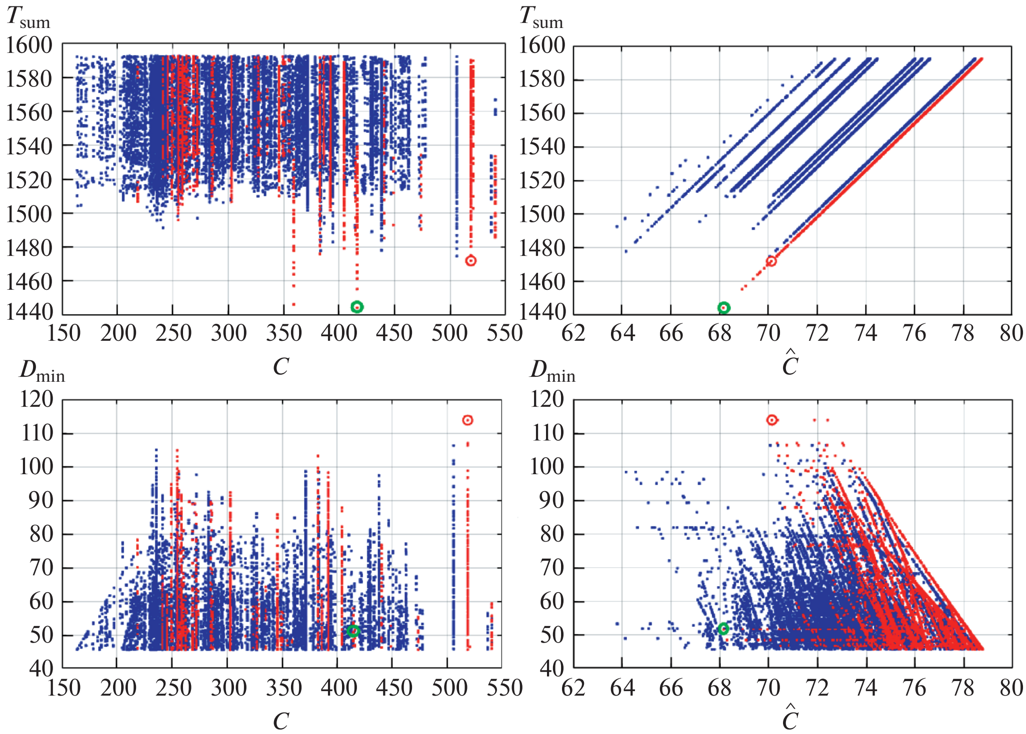

Fig. 2. Execution time of the plan and minimum interception distance for acceptable plans of the same initial situation depending on the complexity and average complexity of the plan.

Further modelling is devoted to investigating plans for a single initial state. Figure 2 presents the dependences $T_{\rm sum}[\pi], D_{\min}[\pi]$ from $C[\pi], \widehat{C}[\pi]$ for acceptable π plans, whose minimum interception distance $D_{\min}[\pi] > 0.6D_{\min}[\pi^*]$, where π^* is the $J_{DT}[\pi]$ optimal plan. The values $T_{\rm sum}[\pi^*], D_{\min}[\pi^*]$ are additionally circled in red, and all points $\{\pi:\ \pi_1 = \pi_1^*\}$ corresponding to acceptable plans are also highlighted in red.

Table 2. Percentage of matches of the first four objectives $\pi_1^*, \pi_2^*, \pi_3^*, \pi_4^*$ of the optimal $J_T[\pi]$ plan π^* with the corresponding dangerous and convenient targets for 1000 different initial state

Target numbers in the plan π^*	Number of matches of i target of the plan π^* with i according to			
	danger $(K_{\pi_i^*} = K_{j_i})$, %	convenience $(U_{\pi_i^*} = U_{j_i})$, %	danger and convenience, %	danger or convenience, %
First target π_1^* $(i = 1)$	65.0	65.9	56.8	74.1
Second target π_2^* $(i = 2)$	32.1	57.6	13.9	75.8
Third target π_3^* $(i = 3)$	19.4	58.5	7.1	70.8
Fourth target π_4^* $(i = 4)$	16.3	57.0	3.8	69.5

Table 3. Percentage of matches of the first four objectives $\pi_1^*, \pi_2^*, \pi_3^*, \pi_4^*$ of the optimal $J_{DT}[\pi]$ plan π^* with the corresponding dangerous and convenient targets for 1000 different initial state

Target numbers in the plan π^*	Number of matches of i target of the plan π^* with i according to			
	danger $(K_{\pi_i^*} = K_{j_i})$, %	convenience $(U_{\pi_i^*} = U_{j_i})$, %	danger and convenience, %	danger or convenience, %
First targe π_1^* $(i = 1)$	61.2	63.0	52.8	71.4
Second targe π_2^* $(i = 2)$	36.6	52.9	13.2	76.3
Third targe π_3^* $(i = 3)$	27.5	49.2	7.7	69.0
Fourth targe π_4^* $(i = 4)$	24.6	45.7	6.2	64.1

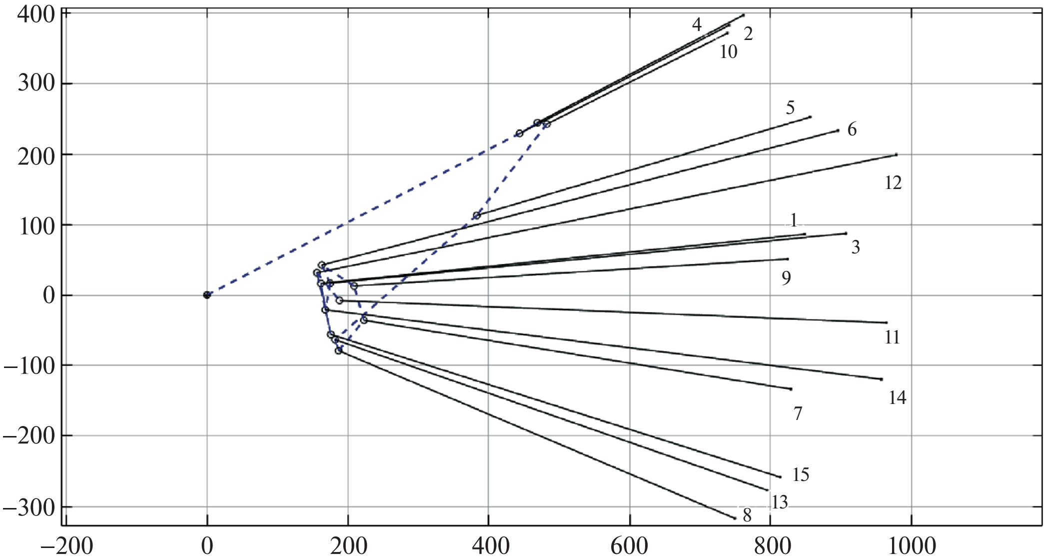

Fig. 3. Optimal interception plan of 15 targets by TS according to the criterion $J_{DT}[\pi] : \pi^* = (2, 4, 10, 5, 13, 8, 7, 9, 6, 3, 14, 1, 15, 12, 11)$, $T_{\text{sum}}[\pi^*] = 1471.049$, $D_{\min}[\pi^*] = 159.168$.

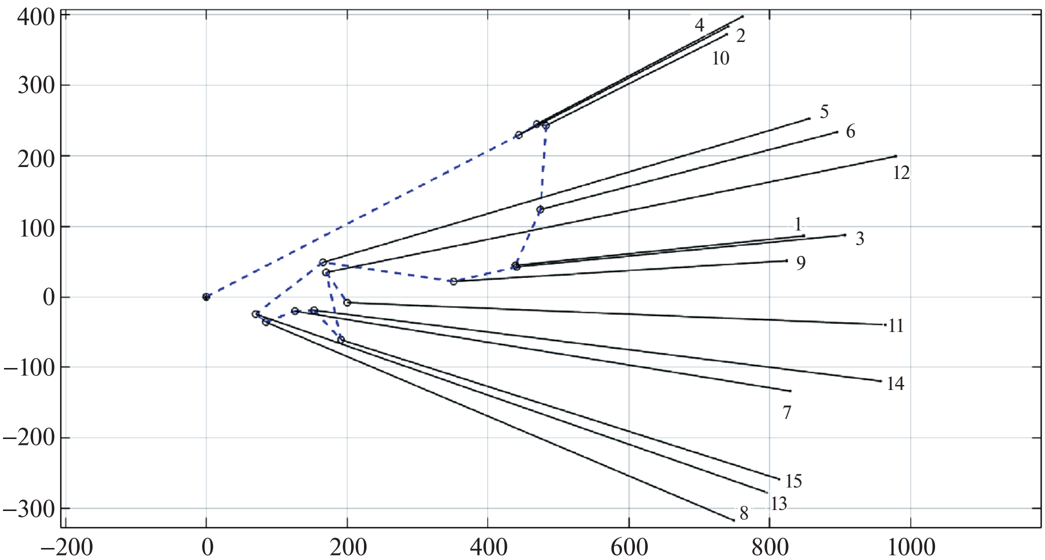

Fig. 4. Optimal interception plan of 15 targets by TS according to the criterion $J_T[\pi] : \pi^* = (2, 4, 10, 6, 1, 3, 9, 5, 13, 8, 7, 14, 15, 12, 11)$, $T_{\text{sum}}[\pi^*] = 1448.051$, $D_{\min}[\pi^*] = 74.183$.

The graph shows that the execution times of all acceptable plans are linearly dependent on $\widehat{C}[\pi]$, which makes it possible to create an optimal polynomial algorithm for constructing an intercept plan in the considered problem. The green circles on all the graphs indicate the optimal J_T plan, the execution time of which is less than $T_{\text{sum}}[\pi^*]$ by 2%, and this plan is worse than π^* by 54% according to the minimum distance of approaching the targets to the origin.

The four acceptable plans $\pi^*, \pi^1, \pi^2, \pi^3$ with the same value $D_{\min} = 113.8$, are analyzed in Fig. 2 on the left, are labeled with a single dot, and on the right they are separated by mean complexity values. Table 4 shows the considered target traverse plans in clear form, where their common part is highlighted.

Table 4. Acceptable plans with $D_{\min}[\pi] = 113.8$

<table>
<tr><th>i</th><th colspan="2">π^i</th><th>$T_{\text{sum}}[\pi]$</th><th>$D_{\min}[\pi]$</th><th>$C[\pi]$</th><th>$\widehat{C}[\pi]$</th></tr>
<tr><td>π^*</td><td rowspan="4">$\{10, 1, 6, 12, 2, 13, 3, 5, 15,$</td><td>8, 11, 9, 4, 14, 7}</td><td>1472</td><td>113.8</td><td>518.8</td><td>70.1</td></tr>
<tr><td>1</td><td>8, 11, 4, 9, 14, 7}</td><td>1496</td><td>113.8</td><td>518.8</td><td>71.9</td></tr>
<tr><td>2</td><td>11, 8, 4, 9, 14, 7}</td><td>1504</td><td>113.8</td><td>518.8</td><td>72.4</td></tr>
<tr><td>3</td><td>11, 8, 9, 4, 14, 7}</td><td>1475</td><td>113.8</td><td>518.8</td><td>70.4</td></tr>
</table>

Table 4 shows that the maximum time between intercepts C and $D_{\min}$ in these plans were achieved in the general plan section. The differing sequences of targets finalizing the plans, however, resulted in a change in the average complexity of each plan.

The modeling section is completed with an example of constructing two optimal intercept plans according to the criteria $J_{DT}[\pi]$ and $J_T[\pi]$ in Figs. 3 and 4 for the same initial state, where the trajectory of TS is highlighted by the blue dashed line.

Optimization according to the criterion $J_T[\pi]$ leads to a slight improvement in the execution time of the plan compared to the optimal $J_{DT}[\pi]$ plan, but at the same time the value of $D_{\min}[\pi]$ is more than halved.

6. CONCLUSION

In this paper the problem of intercepting of a set of rectilinearly moving targets by a single interceptor was considered. New macro characteristics of the problem were proposed and their influence on the construction of the optimal intercept plan for different initial states was statistically investigated. An optimal plan finding algorithm based on intelligent brute-force search and dynamic programming concepts was proposed to collect statistics in adequate time. The impact of the new quantities on mission success is shown on the collected statistics and conclusions are made about its applicability to the creation of fast greedy intercept algorithms.

There are plans to investigate various local rules that take into account the state information and geometric characteristics of the target distribution, build suboptimal intercept algorithms based on them, and compare them with the brute-force optimal algorithms.

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